Calculus III Honors/MAT397 Summer Packet

SCHOOL YEAR 2023-2024

Welcome to MAT397! This summer you'll be expected to read through and complete the following three assignments on separate sheets of paper:

Topic	Recommended Due Date
Single Variable Calculus Review/Extension Questions	8/10/23
Determinants of Matrices	8/17/23
Integrating Trigonometric Functions	8/17/23
Introduction to Vectors	8/24/23

All work must be shown for credit, and these concepts will be integral (pun intended) to the work we will be doing this coming school year.

I wish you luck and am excited for a great year ahead!

- Mr. Greg Aschoff

P.S.: If you did not earn at least a 4 on the BC exam, you **must** contact me via e-mail at gaschoff@westex.org as soon as possible if you wish to still register for credit with Syracuse University. You **may** be eligible for an alternative test to be administered in September at the discretion of the SUPA Course Supervisors.

Determinants & Inverse Matrices

The *determinant* of the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the number $ad - cb$.

The above sentence is abbreviated as

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

Example.

$$\det \begin{pmatrix} 4 & -2 \\ 1 & -3 \end{pmatrix} = 4(-3) - 1(-2) = -12 + 2 = -10$$

The determinant of a 3×3 matrix can be found using the formula

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Example.

$$\det \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 0 & 1 \end{pmatrix} = 2 \det \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} - (-1) \det \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} + 0 \det \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$$

$$= 2[3 \cdot 1 - 0(-2)] + [0 \cdot 1 - 1(-2)] + 0$$

$$= 2 \cdot 3 + 2$$

$$= 8$$

Exercises

For #1-6, compute the determinant of the given matrix.

1.)

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

2.)

$$\begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix}$$

3.)

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$$

4.)

$$\begin{pmatrix} 3 & 0 & 0 \\ 107 & 1 & 0 \\ \sqrt{2} & 2 & 6 \end{pmatrix}$$

5.)

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$$

6.)

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix}$$

Integrating Trigonometric Functions

1. (a) Find $\displaystyle\int \cos^3 x \, dx$ (b) $\displaystyle\int \cos^5 x \, dx$ (c) $\displaystyle\int \sin^5 x \cos^2 x \, dx$.

2. Evaluate $\int \sin^2 x \cos^2 x dx$ by using the double angle formulae

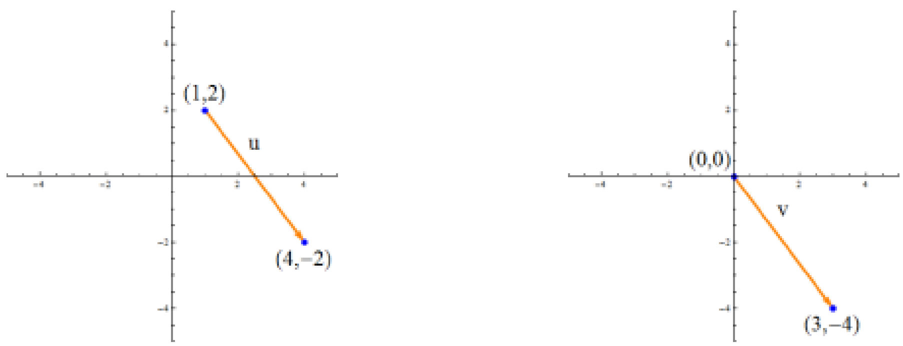
$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

3. Using the double angle formulae twice find $\displaystyle\int \sin^4 x \cos^2 x \, dx$.

Vector Geometry in Two and Three Dimensions

Vectors in Two Dimensions

You've probably heard of vectors as objects with both magnitude and direction, or something along these lines. Another way to envision a vector is as an arrow from one point to another. A vector starts at some **basepoint** and extends to some **terminal point**. For instance, Figure 1a shows the vector $\mathbf{u}$ with basepoint $(1, 2)$ and terminal point $(4, -2)$.



(a) The vector from $(1, 2)$ to $(4, -2)$. (b) The vector from $(0, 0)$ to $(3, -4)$.

Figure 1: Vectors as arrows.

How does this notion of a vector as an arrow from one point to another fit with the magnitude-and-direction description? The magnitude of the vector $\mathbf{u}$ in Figure 1a is simply the **length** of the arrow, which we can compute using the distance formula in the plane. The distance from the basepoint $(1, 2)$ to the terminal point $(4, -2)$ is given by

$$\text{distance} = \sqrt{(1-4)^2 + (2-(-2))^2} = \sqrt{9+16} = 5,$$

so the length (or magnitude) of $\mathbf{u}$ is 5. We typically denote the length of a vector with double bars: $\|\mathbf{u}\| = 5$. The vector $\mathbf{u}$ also has a direction — it's pointing in the direction of $(4, -2)$ from $(1, 2)$ — so $\mathbf{u}$ can be described as the vector based at $(1, 2)$ pointing in the direction of $(4, -2)$ with magnitude 5. This description is not very pleasant, but we'll try to rectify this.

Notice that the vector $\mathbf{v}$ in Figure 1b has the same magnitude and direction as $\mathbf{u}$, but has a different basepoint — it's just a translated copy of $\mathbf{u}$. But if we describe the direction of $\mathbf{v}$ as we did the direction of $\mathbf{u}$, we would say that $\mathbf{v}$ is the vector based at $(0, 0)$ pointing in the direction of $(3, -4)$. From their descriptions, it's not immediately evident that $\mathbf{u}$ and $\mathbf{v}$ are equivalent vectors. One way to make comparisons of $\mathbf{u}$ and $\mathbf{v}$ a bit clearer is to describe them not by their base and terminal points, but by the displacement that they cause. Each vector moves 3 units right and 4 units down from its respective base points, so we can write $\mathbf{u} = \langle 3, -4 \rangle$ and $\mathbf{v} = \langle 3, -4 \rangle$, with basepoints understood. We call 3 and -4 the $\mathbf{x}-$ and $\mathbf{y}-$

components, respectively.

Components of a vector. Let $\mathbf{v}$ be the vector with basepoint $P = (x_1, y_1)$ and terminal point $Q = (x_2, y_2)$. Then the **x-component** of $\mathbf{v}$ is $v_1 = x_2 - x_1$ and the **y-component** of $\mathbf{v}$ is $v_2 = y_2 - y_1$. We may then write $\mathbf{v} = \langle v_1, v_2 \rangle$. The **zero vector** is the vector whose $x-$ and $y-$components are both zero: $\mathbf{0} = \langle 0, 0 \rangle$.

Once we've written a vector in terms of its components, we can easily compute the its length:

$$\|\mathbf{v}\| = \|\langle v_1, v_2 \rangle\| = \sqrt{v_1^2 + v_2^2}.$$

Example 1.1. Determine whether or not the vector $\mathbf{u}$ from $(2, 6)$ to $(7, 18)$ is equivalent to the vector $\mathbf{v}$ from $(-1, 4)$ to $(4, 15)$. Calculate the length of $\mathbf{u}$.

(*Solution*) Let's write $\mathbf{u}$ and $\mathbf{v}$ in terms of their components:

$$\mathbf{u} = \langle 7 - 2, 18 - 6 \rangle = \langle 5, 12 \rangle, \quad \text{and} \quad \mathbf{v} = \langle 4 - (-1), 15 - 4 \rangle = \langle 5, 11 \rangle.$$

With this description of $\mathbf{u}$ and $\mathbf{v}$, it's easy to see that they are not equivalent. We can also compute the length of $\mathbf{u}$ from this components description:

$$\|\mathbf{u}\| = \sqrt{5^2 + 12^2} = 13.$$

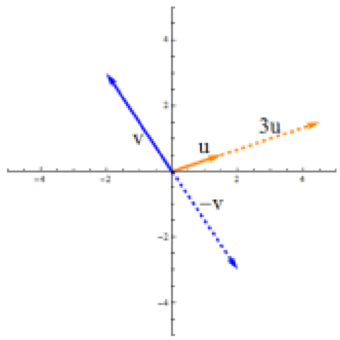
◊

One of the many advantages to the component notation for vectors is that it makes vector algebra very easy: we can just do all operations component-wise. For instance, we'd like to define **scalar multiplication** on vectors, which is the operation of stretching a vector by some real factor. Figure 2a depicts the scaling of a vector $\mathbf{u}$ by a factor of 3 — resulting in a vector pointing in the same direction, but with magnitude three times that of $\mathbf{u}$ — and the scaling of a vector $\mathbf{v}$ by -1, which results in a vector with the same magnitude as $\mathbf{v}$, but pointing in the opposite direction. Component notation helps us here because we can scale a vector component-wise. If we wrote a vector in terms of its basepoint and terminal point, then writing down its scaled version would require finding the terminal point of the scaled version. But component notation just gives the *displacement* corresponding to the vector, and we can obtain the scaled vector by simply scaling the displacement. If $\mathbf{v} = \langle v_1, v_2 \rangle$ is a vector and $\lambda \in \mathbf{R}$ is a scalar, then the scalar multiple $\lambda\mathbf{v}$ of $\mathbf{v}$ is given by

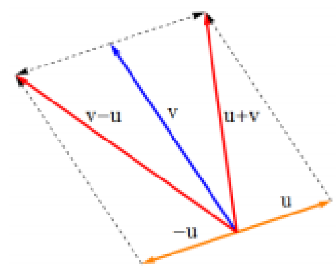
$$\lambda\mathbf{v} = \langle \lambda v_1, \lambda v_2 \rangle.$$

We immediately notice that

$$\|\lambda\mathbf{v}\| = \sqrt{(\lambda v_1)^2 + (\lambda v_2)^2} = \sqrt{\lambda^2(v_1^2 + v_2^2)} = |\lambda|\sqrt{v_1^2 + v_2^2} = |\lambda|\|\mathbf{v}\|,$$



(a) Scalar multiplication of vectors.



(b) Addition of vectors.

Figure 2: Vector operations.

exactly as we'd expect. If one vector is a scalar multiple of another, we say that the two vectors are **parallel**, since they point in the same direction.

We can similarly define addition and subtraction of vectors. If we have vectors $\mathbf{u}$ and $\mathbf{v}$, then their sum is the vector obtained by following one of the vectors and then the other, as in Figure 2b. That is, the displacement associated to $\mathbf{u}+\mathbf{v}$ is the sum of the displacements associated to $\mathbf{u}$ and $\mathbf{v}$, respectively:

$$\mathbf{u}+\mathbf{v} = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1+v_1, u_2+v_2\rangle.$$

From Figure 2b we see that vector addition must be commutative: $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$. We can then define the difference $\mathbf{v}-\mathbf{u}$ to be the result of adding $-\mathbf{u}$ to $\mathbf{v}$:

$$\mathbf{v}-\mathbf{u} = \langle v_1, v_2\rangle + \langle -u_1, -u_2\rangle = \langle v_1-u_1, v_2-u_2\rangle.$$

You can easily check that the zero vector behaves as we'd like (meaning that $\mathbf{0}+\mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$), and that since our vector operations are defined component-wise, they inherit associativity and distributivity from their usual counterparts.

Vectors in Three Dimensions

We've warmed up to vectors by considering them in two dimensions, but in this class most of our time will be spent in (at least) three dimensions. Thankfully, almost all of our considerations in two dimensions carry over to three dimensions quite easily. As before, a vector can be thought of as an arrow from a basepoint to a terminal point, but we typically think of a vector as an object with three **components**:

$$\mathbf{v} = \langle a, b, c\rangle.$$

We call these the x-, y-, and z-components of $\mathbf{v}$, respectively. Vector addition and scalar multiplication work exactly as before, and we can still obtain a unit vector pointing in the direction of $\mathbf{v}$ by scaling $\mathbf{v}$ by a factor of $\|\mathbf{v}\|^{-1}$. We can no longer determine a unit vector by a single angle θ, but a pair of angles θ and ϕ will determine an angle via *spherical coordinates*. We won't worry about this here, though.

We will consider two items in three dimensions that we omitted in two dimensions. The first is a bit of notation. We call the vectors

$$\mathbf{i} = \langle 1, 0, 0\rangle, \quad \mathbf{j} = \langle 0, 1, 0\rangle, \quad \mathbf{k} = \langle 0, 0, 1\rangle$$

the **standard basis vectors** in $\mathbf{R}^3$. Our primary use for this notation comes from the fact that every vector in $\mathbf{R}^3$ can be written uniquely as a linear combination of the standard basis vectors:

$$\langle a, b, c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

The second item we'll consider is how to describe lines in $\mathbf{R}^3$, which we'll do using **parametric equations**. In any dimension, a line is determined by choosing a point through which the line should pass and a direction the line should travel. In $\mathbf{R}^2$ this is often expressed via the *point-slope form* for equations of lines. An expression that's more amenable to three dimensions describes a line by giving a point and a **direction vector**.

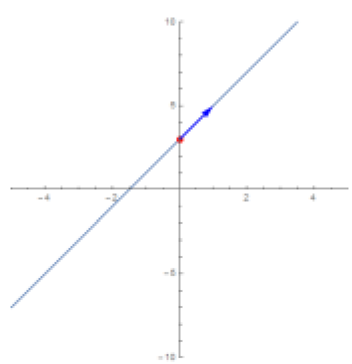


Figure 4: A line in $\mathbf{R}^2$.

For instance, Figure 4 shows the line whose slope-intersect equation is $y = 2x + 3$. This line passes through the point $(0, 3)$ and leaves $(0, 3)$ heading in the direction $\langle 1, 2\rangle$, because as x increases by 1 unit, y increases by 2. So we can give another description of this line: it's the line traced out by

$$\mathbf{r}(t) = \langle 0, 3\rangle + t\langle 1, 2\rangle, \tag{1}$$

where we let t vary over all real numbers. Notice that when $t = 0$, $\mathbf{r}(t)$ is at our point $(0, 3)$. We can also describe our line in terms of its components, each of which is a function of t:

$$x(t) = t, \qquad y(t) = 2t + 3. \tag{2}$$

These two descriptions — vector parametrizations such as (1) and parametric equations such as (2) — will be how we describe lines in three space.

Equation of a Line. The line in $\mathbf{R}^3$ that passes through the point (x_0, y_0, z_0) in the direction of $\mathbf{v} = \langle a, b, c\rangle$ is described by

$$\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

and

$$x(t) = x_0 + at, \qquad y(t) = y_0 + bt, \qquad z(t) = z_0 + ct,$$

where we allow t to take on all real values. Notice that any vector parallel to $\mathbf{v}$ will determine the same line.

Example 1.4. (§13.2, Exercise 37 of [1]) Find a vector parametrization for the line that passes through $(1, 1, 1)$ and $(3, -5, 2)$.

(*Solution*) Since our line passes through $(1, 1, 1)$, we may parametrize it as

$$\mathbf{r}(t) = \langle 1, 1, 1\rangle + t\mathbf{v}$$

for some direction vector $\mathbf{v}$. Because the line also passes through $(3, -5, 2)$, it must be parallel to the vector whose basepoint is $(1, 1, 1)$ and whose terminal point is $(3, -5, 2)$, so we might as well choose this to be our direction vector:

$$\mathbf{v} = \langle 3 - 1, -5 - 1, 2 - 1\rangle = \langle 2, -6, 1\rangle.$$

So the line has vector parametrization

$$\mathbf{r}(t) = \langle 1, 1, 1\rangle + t\langle 2, -6, 1\rangle.$$

Notice that any scalar multiple of $\mathbf{v}$ would have also worked as a direction vector. ◊

Example 1.5. (§13.2, Exercise 53 of [1]) Determine whether the lines $\mathbf{r}_1(t) = \langle 0,1,1\rangle + t\langle 1,1,2\rangle$ and $r_2(s) = \langle 2,0,3\rangle + s\langle 1,4,4\rangle$ intersect, and if so, find the point of intersection.

(*Solution*) Suppose we have numbers t and s so that $\mathbf{r}_1(t) = \mathbf{r}_2(s)$. (Notice that t and s are not necessarily the same number.) Then

$$\mathbf{r}_1(t) = \langle 0,1,1\rangle + t\langle 1,1,2\rangle = \langle 2,0,3\rangle + s\langle 1,4,4\rangle = \mathbf{r}_2(s).$$

Since $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ must agree component-wise, we see that t and s must give a solution to the following system of linear equations:

$$\begin{aligned} t &= s+2 \\ t+1 &= 4s \\ 2t+1 &= 4s+3 \end{aligned} .$$

We can substitute the first equation into the second to arrive at $s+3=4s$. This is easily solved to find that $s=1$, which then implies that $t=1+2=3$. Because $s=1$ and $t=3$ also satisfies the final equation, these parameter values give a solution to the system. So we have

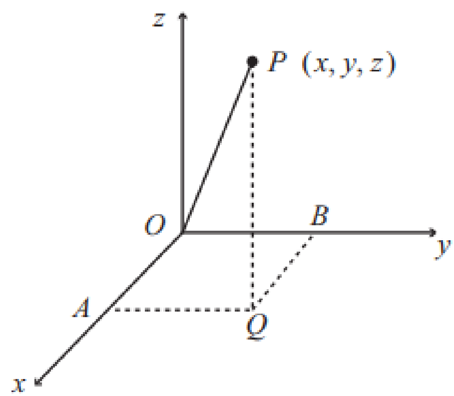
$$\mathbf{r}_1(3) = \langle 4,5,7\rangle = \mathbf{r}_2(1),$$

and indeed the two lines intersect. ◊

The length of a position vector

What is the length of the position vector $\overline{OP}$?

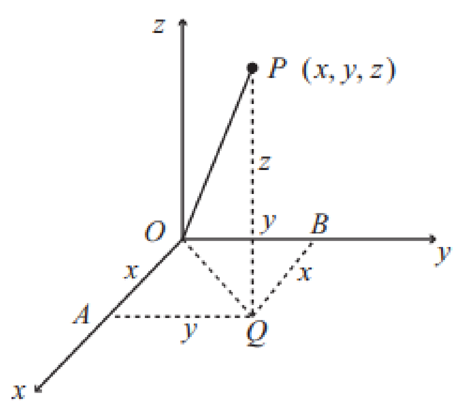
To answer this question, we start by dropping a perpendicular from P down to the (x, y)-plane. We shall call this new point Q. Then we join the point Q up to the x and y axes, again at right angles. We shall call the two new points A and B.



Now we know some of the lengths in this diagram. First, the length PQ is the height of the point P above the (x, y)-plane. So that length must be z.

The length OA is the distance along the x coordinate axis, so that length must be x. And the length BQ is the same as the length OA, so that must also be x.

In the same way, the length OB is the distance along the y coordinate axis, so that length must be y. And the length AQ is the same as the length OB, so that must also be y.



Now we join the points O and Q. Then OAQ is a right-angled triangle, and so is OBQ. So the length OQ can be found by using Pythagoras's Theorem, in either of these triangles. We obtain the formula

$$
\begin{aligned}
OQ &= \sqrt{OA^2 + AQ^2} \qquad (\text{or} \quad \sqrt{OB^2 + BQ^2}) \\
&= \sqrt{x^2 + y^2}.
\end{aligned}
$$

Now we can use the right-angled triangle OQP. If we apply Pythagoras's Theorem to this triangle, we obtain

$$
\begin{aligned}
OP &= \sqrt{OQ^2 + QP^2} \\
&= \sqrt{\left(\sqrt{x^2 + y^2}\right)^2 + z^2} \\
&= \sqrt{x^2 + y^2 + z^2}.
\end{aligned}
$$

Key Point

If P is the point with coordinates (x, y, z) then the length, or magnitude, of the position vector $\overline{OP}$ is given by the formula

$$|\overline{OP}| = OP = \sqrt{x^2 + y^2 + z^2}.$$

Exercises

1. Find the lengths of each of the following vectors
   (a) $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$
   (b) $5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$
   (c) $2\hat{\mathbf{j}} - \hat{\mathbf{k}}$
   (d) $5\hat{\mathbf{i}}$
   (e) $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$
   (f) $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$
2. Determine the vector $\overline{AB}$ for each of the following pairs of points
   (a) A (3,7,2) and B (9,12,5)
   (b) A (4,1,0) and B (3,4,-2)
   (c) A (9,3,-2) and B (1,3,4)
   (d) A (0,1,2) and B (-2,1,2)
   (e) A (4,3,2) and B (10,9,8)

3. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v} = \langle 1,3\rangle$ and $\mathbf{w} = \langle -1,-5\rangle$. ⇒

4. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v} = \langle 1,2,3\rangle$ and $\mathbf{w} = \langle -1,2,-3\rangle$. ⇒

5. Let $P = (4,5,6)$, $Q = (1,2,-5)$. Find $P\acute{Q}$. Find a vector with the same direction as $P\acute{Q}$ but with length 1. Find a vector with the same direction as $\overrightarrow{PQ}$ but with length 4. ⇒
Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v} = \langle 3,2,1\rangle$ and $\mathbf{w} = \langle -1,-1,-1\rangle$. ⇒

6. If A, B, and C are three points, find $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$. ⇒

7. Consider the 12 vectors that have their tails at the center of a clock and their respective heads at each of the 12 digits. What is the sum of these vectors? What if we remove the vector corresponding to 4 o'clock? What if, instead, all vectors have their tails at 12 o'clock, and their heads on the remaining digits? ⇒